THE DEVELOPMENT OF UNITIZING: ITS ROLE IN CHILDREN’S PARTITIONING STRATEGIES

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The goal of this study was to gain a better understanding of the intuitive activity that precedes the construction of quantity, in particular, to make inferences concerning developmental trends and needs in children’s unitizing processes. Using a cross-sectional design, the study analyzed the partitioning strategies of 346 children from grades four through eight in terms of a framework that translated economy in the number or size of pieces and the use of perceptual cues into sophistication in unitizing. At each grade level, a greater percentage of students used economical partitioning strategies than used less economical cut-and-distribute strategies. As grade level increased, the percentage of students using economical strategies increased, indicating a shift away from the distribution of singleton units toward the use of more composite units. Strategies were heavily influenced by social practice related to the commodity being shared and, to a lesser degree, by the numerical portion of the given extensive quantities.

BACKGROUND

Research in the nonsymbolic representation of fractions (Pothier & Sawada, 1983, 1984; Hunting, 1983; Kieren, Nelson, & Smith, 1985) highlights the need for children to build a deep understanding of fractions by using a variety of concrete and pictorial models. In particular, partitioning activities are important mechanisms for building rational number understandings (Piaget, Inhelder, & Szeminska, 1960; Pothier, 1981; Kieren, 1976, 1980; Streefland, 1991). The ability to divide an object or a set of objects into equal parts appears critical to the logical development of part-part and part-whole relationships and notions of equality and inequality, and it may influence children’s understanding of other mathematical topics such as measurement and geometry (Pothier & Sawada, 1990).

In the last decade, other strands of research have turned toward investigating the role of the mathematics of quantity (Schwartz, 1976, 1988; Kaput, 1985) in the development of rational number concepts. This perspective emphasizes the critical linkage of numbers and their referents, that is, the linkage of units of measure and magnitude of quantities, for understanding relations and operations. There is a consensus that as one encounters the domain of rational numbers, changes in the nature of the unit largely account for the cognitive complexity entailed in linking meaning, symbols, and operations (Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988).

Unitizing is the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity. For example, given a case of cola, one could think of it as 24 (cans or 1-units), 2 (12-packs), or 4 (six-packs). The ability to form and operate with increasingly complex unit structures appears to be an important mechanism by which more sophisticated reasoning develops. Research in proportional reasoning, for example, indicates that one of the most salient differences between proportional reasoners
and nonproportional reasoners is that the proportional reasoners are adept at building and using composite extensive units and that they make decisions about which unit to use when choices are available, choosing more composite units when they are more efficient than using singleton units (Lamon, 1993a, 1993b).

The perspective that the ability to conceive of quantities in terms of increasingly complex units builds mathematical reasoning power has a sound theoretical basis in Piaget’s operational level of thinking (Piaget, 1969). Furthermore, this perspective has been demonstrated in several mathematical domains, including the acquisition of early counting strategies (Steffe, Cobb, & von Glasersfeld, 1988), the acquisition of early addition and subtraction strategies (Carpenter & Moser, 1983), the development of the concepts of multiplication and division (Steffe, 1988), and the development of proportional reasoning (Lamon, 1994). Research on the natural development of language hierarchies (Callanan & Markman, 1982; Markman, 1979) further substantiates this perspective, suggesting that more sophisticated thinking results when one reframes a situation in terms of a more collective unit because this allows the student to think about both the aggregate and the individual items that compose it.

Together, the bodies of research concerning unitizing and partitioning suggest that the two processes build different and essential perspectives toward the understanding of rational numbers. Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of constructing rational numbers to a child’s informal knowledge about fair sharing. Unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during, and after the sharing process. To understand the process by which children come to know rational numbers as entities unto themselves, we need to better understand the interaction of intuitive activity and cognitive processes that precedes the construction of a rational number quantity.

Goals and Rationale

The purpose of the study reported in this article was to make children’s tacit unitizing process explicit through partitioning activities, to define more clearly the relationship between the two processes, and to connect children’s partitioning and unitizing strategies to given contexts. By taking snapshots of children’s partitioning strategies at each grade level, fourth through eighth, it also sought to identify trends or stages that characterize the development of increasingly sophisticated unitizing ability. Pothier and Sawada (1983) proposed a five-level theory of the development of the partitioning process based on their work with children in grades K–3. According to their theory, young children first employ a halving mechanism in the process of sharing. The second level of the partitioning process is marked by an algorithmic application of the doubling process to produce fractional parts whose denominators are powers of 2. While in the first two stages children are merely concerned with counting pieces; in the third stage, they focus on a dual notion of evenness: the equality or sameness of the parts they are producing, and the ability to represent fractional parts whose denominators are even numbers. In the fourth stage, the children overcome the limitations of the halving algorithm and adopt a new cutting and counting procedure to produce fractional parts whose denominators are odd.
In the fifth stage, a child observes that, for example, ninths can be produced by trisecting thirds and begins to use a multiplicative algorithm to partition composite numbers. Although none of the children in the age group examined by Pothier and Sawada used the multiplicative strategy, it represents a more efficient alternative to the stage four cutting and counting algorithm and may be readily observed in older children.

Children in the stage of concrete operations (Piaget, Inhelder, & Szeminska, 1960) exhibit grouping operations and, because processes become reversible, can flexibly put things together to form classes and separate collections into subgroups. They begin to adopt multiple perspectives and to make decisions, usually based on economy or efficiency, about which units to use when alternatives are available (Lamon, 1993a, 1993b). Thus, the stage 5 thinking described in Pothier and Sawada’s (1983) partitioning theory is characteristic of children in the concrete operations stage. It is likely that after some initial exploratory phase, in which students become comfortable with fractional parts, their multiplicative algorithm for producing ninths starting with thirds is an efficiency resulting from observations about equivalence.

The hypothesis underlying this research is that the mathematical power afforded by the notion of equivalence should, for concrete operational students, also work to help them to compose units. Growth in sophistication of the unitizing process, signified by the use of more composite units or larger units, should be reflected in students’ partitioning processes. After the partitioning stages characterized by Pothier and Sawada in terms of the ability to produce finer and finer partitions, or, perhaps, overlapping with their fifth stage, additional stages should emerge in which partitioning strategies grow increasingly economical.

**METHOD**

**Procedure**

Children from grades 4 through 8 were given 11 tasks in which they were asked to draw pictures to show how they would share various types of food among given numbers of people. Five children were chosen at random from each grade level to participate in standardized clinical interviews. They were given the same tasks in the same written format as the rest of their classmates but were asked to think out loud while they were working so that the researcher could gain further insights into their methods and reasoning. These children were asked to explain what they were doing and why, to decide if there was a better way to share the food, and to say how much of the food one person would receive.

**Sample**

The 346 students who participated in the study came from three schools in two midwestern cities. Three intact classes from each grade level, grades four through eight (n$_4$ = 63, n$_5$ = 60, n$_6$ = 72, n$_7$ = 69, n$_8$ = 82) from one parochial school (K–8) and two public schools (K–6 and 6–8) participated in the study. Each class was heterogeneously grouped and culturally diverse. All teachers reported that they had never given their students partitioning activities.
Tasks

The partitioning tasks used in this study are described in Table 1. The tasks were designed to include several known and hypothesized influences on children’s partitioning activities. In addition to the usual distinction between discrete and continuous elements, tasks were differentiated according to the manner in which food items are packaged. For example, the packaging of food items may suggest or constrain certain partitionings, as in the case of eggs packaged in standard arrays in cartons, and sectioned candy bars. Composite units such as six-packs of cola or packs of gum may be left intact or opened to reveal individual items. Differences in partitioning were also expected when the items to be shared were alike, such as several pizzas all with the same topping, as opposed to sharing pizzas with different toppings or several different Chinese dinners.

Framework for Analysis

The notation developed by Behr, Harel, Post, & Lesh (1992), hereafter referred to as the Behr notation, combines a pictorial representation of children’s operations on the objects given in the partitioning task and a symbolic mathematical representation of the quantities and relationships involved in the partitioning task. Whereas the Behr et al. (1992) semantic analysis represents idealized models of partitive divisions, children’s actual partitioning activity often involves other details. For example, to divide three pies among four people, some children cut each of the three 1-units into eight parts and then distribute the pieces. Sometimes there are differences in the way children mark fractional parts on the objects and in the way they actually cut them. For example, when the whole consists of four 1-units to be partitioned into three shares, each of the 1-units may be marked as sixths and cut into thirds, or in three of the 1-units, halves may be marked, and only the fourth 1-unit might be marked and cut into thirds. This study is concerned with some of those finer details because they provide insight into the sophistication of children’s unitizing process and their dependence on perceptual supports. To account for these distinctions, partitioning is defined in this study as determination of equal shares and is viewed as a multistage operation: marking objects, cutting them, and clearly indicating one person’s share.

Pilot studies were used to classify children’s strategies for fair sharing. These studies revealed that strategies may be differentiated along at least four dimensions: (a) preservation of pieces that did not require cutting in cases where each person receives more than one object in a discrete quantity; (b) economy of the marking (not using sixths when thirds suffice); (c) economy of cutting (not making more cuts than necessary); and (d) the nature, packaging, and social practices related to the objects being shared. In general, a higher level of sophistication in the unitizing process is indicated by the ability to use more composite units, and in the context of these activities, this is indicated by the preservation of pieces that do not require cutting and economy in making necessary marks and cuts. However, an exception occurs when customary practice supersedes concerns for economical marking and cutting, as would be the case when sharing three pizzas, each with different toppings, among three people. The obvious and most economical way to share is to give each person one pizza, but it is customary for each person to have some of each pizza.
Table 1
**Partitioning Tasks Used to Study Unitizing**

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Identifier</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td>eggs</td>
<td>You have the carton of 12 eggs pictured below, and 3 people who want to eat them for breakfast.</td>
</tr>
<tr>
<td>Subsets separable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Array form</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discrete</td>
<td>gum</td>
<td>You have 5 packs of gum and 4 people. (A pack of gum has 5 sticks of gum inside.)</td>
</tr>
<tr>
<td>Subsets separable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Composite form</td>
<td>cola</td>
<td>You have 8 six-packs of cola and 3 people.</td>
</tr>
<tr>
<td>Continuous</td>
<td>pepperoni</td>
<td>You have 4 pepperoni pizza pies and 3 people.</td>
</tr>
<tr>
<td>Elements dissectible</td>
<td>pizza</td>
<td></td>
</tr>
<tr>
<td>Like items</td>
<td>chocolate</td>
<td>You have 4 chocolate chip cookies and 3 children.</td>
</tr>
<tr>
<td></td>
<td>chip cookies</td>
<td></td>
</tr>
<tr>
<td>Continuous</td>
<td>oatmeal</td>
<td>You have 4 oatmeal cookies and 6 children.</td>
</tr>
<tr>
<td>Elements dissectible</td>
<td>cookies</td>
<td></td>
</tr>
<tr>
<td>Unlike items</td>
<td>4 pizzas</td>
<td>You have 1 cheese pizza, 1 mushroom pizza, 1 sausage pizza, and 1 pepperoni pizza for 3 people.</td>
</tr>
<tr>
<td></td>
<td>3 meals</td>
<td>You have 3 Chinese dinners (1 pork, 1 beef, and 1 chicken) and 6 people to eat dinner.</td>
</tr>
<tr>
<td></td>
<td>4 meals</td>
<td>You have 4 Chinese dinners (1 pork, 1 beef, 1 chicken, and 1 seafood) and 3 people for dinner.</td>
</tr>
<tr>
<td>Continuous</td>
<td>candy</td>
<td>You have the 2 candy bars shown below and 5 children.</td>
</tr>
<tr>
<td>Subsets separable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prepartitioned</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The following framework analyzes children’s partitioning strategies in terms of markings and cuts and uses Behr notation to elaborate the quantities and relationships involved for partitioning wholes that consist of multiple, continuous, dissectible objects:

**Share four like pizzas among three people.**

Analogous strategies for other types of wholes will be briefly described, but not elaborated. The Behr notation justifies the ranking of the strategies according to decreasing sophistication in unitizing by providing a powerful pictorial and symbolic representation of the increasing fragmentation of a share of the pizza.
Strategy 1: Preserved-pieces strategy (PP). When each person is to receive more than one 1-unit of the total quantity being shared, the student marks and cuts only the piece that requires cutting and leaves the 1-unit unmarked and intact. This strategy is analyzed in Figure 1.

Children use analogous strategies when partitioning discrete quantities in composite form. For example, when a share includes an entire pack of gum or an entire
six-pack of cola, that pack of gum or pack of cans will be drawn as a composite unit, without indicating the individual units that it contains.

**Strategy 2: Mark-all strategy (MA).** In this strategy, all of the pieces are marked, even those that will remain intact, but only the piece(s) that require cutting will be cut. This strategy is analyzed in Figure 2.

In the case of discrete quantities in composite form, students using the mark-all strategy, for example, will show the six individual cans in a six-pack of cola, or draw the five sticks of gum inside a package.

\[\text{Four 1-units} \quad \begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\text{Three 1-units} & [?] & [?] & [?]
\end{array}\]

Each of the 1-units is partitioned into three parts:

\[\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

\[\begin{array}{ccc}
[?] & [?] & [?]
\end{array}\]

Each of the three parts of the 1-units is reunitized as one $\frac{1}{3}$-unit to give three $\frac{1}{3}$-units:

\[\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

\[\begin{array}{ccc}
[?] & [?] & [?]
\end{array}\]

The three partitioned 1-units and the three $\frac{1}{3}$-units are each distributed equally among the three 1-units:

\[\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

\[\begin{array}{ccc}
[?] & [?] & [?]
\end{array}\]

*Figure 2. The mark-all strategy.*

**Strategy 3: Distribution strategy (D).** All pieces of the whole are marked and cut, and the smaller pieces are distributed. The analysis of this strategy appears in Figure 3.

For wholes consisting of discrete quantities with subsets separable, a student using the distribution strategy, for example, will separate the squares of a candy bar or
the cans in a six-pack of cola into subsets consisting of single elements or small chunks and then distribute them.

The marking used in each of the three basic strategies may or may not be economical. For example, a student might choose to mark a pizza with six equal pieces when sharing among three people or might mark a candy bar into two-piece chunks when four-piece chunks could be used. This results in the additional classification of each of the partitioning strategies as

EM for economically marked or
OM for overly-marked.

Four 1-units

Three 1-units

Each of the 1-units is partitioned into three parts:

Each of the three parts in each 1-unit is reunitized as one
1/3-unit to give three 1/3-units:

Each set of three 1/3-units is distributed equally among the three 1-units:

For the distribution strategy, pieces were always cut as marked, but in the preserved-pieces strategy and the mark-all strategy, cutting was not always executed in agreement
with the marking. Thus, even when a student has over-marked, say, into six parts for three people, the pieces may be cut two together, as if the object had been marked as thirds. PP and MA strategies may then be classified as

CE for cut more economically than marked or CM for cut as marked.

Another category labeled “In” was created for strategies that were incomplete or incomprehensible or that resulted in invalid partitions. The combinations of characteristics described above form nine strategy types that may be arranged into a hierarchy as shown in Figure 4. Children’s partitioning strategies were coded 1–9 to correspond to these combinations of descriptors. After a brief training session, two mathematics graduate students coded a random sample of 10 student papers (110 responses) to check the reliability of the coding scheme. There was interjudge agreement on 94% of the student strategies.

Figure 4. Hierarchy of partitioning.
RESULTS

Partitioning Wholes Consisting of Continuous, Dissectible, Like Elements

Table 2 shows the percentages of children at each grade level using each of the partitioning strategies on composite wholes consisting of continuous, dissectible, like elements. Figure 5 illustrates each of the strategies using student productions for the pepperoni pizza problem (sharing four pepperoni pizza pies among three people). Between grade levels, the data show trends toward economy from grade 5 to grade 8, and within grade, a preference for economical marking and for the preserved pieces strategy. Although the problems involving pizza pies and chocolate chip cookies involved the same numbers (four items for three people), there was a stronger tendency to use the preserved pieces strategy when partitioning the cookies. Likewise, in the case when four oatmeal cookies were shared among six people, thus requiring a distribution strategy, students tended to minimize the marking and cutting. During his interview, Jason explained the need to be more economical when sharing cookies:

I: Do you see any connection between the pizza pie problem and the chocolate chip cookies problem?
J: Well, they both have four foods and three kids.
I: But I noticed that you did not cut up the food the same way in both situations.
J: When you have to share cookies you have to be more careful because when you cut 'em, the little crumbs break off and go all over and then you have less cookie. When you cut the pizza, you don’t have to worry 'cause it doesn’t get wasted on the floor.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
<th>Preserved-pieces strategy</th>
<th>Mark-all strategy</th>
<th>Distribution strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>EM-CM</td>
<td>OM-CE</td>
<td>OM-CM</td>
</tr>
<tr>
<td>Pizza</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>15</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>25</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>44</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>45</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Chips</td>
<td>4</td>
<td>24</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>25</td>
<td>5</td>
<td>5</td>
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<td></td>
<td>6</td>
<td>38</td>
<td>12</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>58</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>66</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Oatmeal</td>
<td>4</td>
<td>4</td>
<td>33</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>43</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>49</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>68</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>72</td>
<td>13</td>
<td>15</td>
</tr>
</tbody>
</table>

n₁ = 63; n₂ = 60; n₃ = 72; n₄ = 69; n₅ = 82
In = incomplete, incomprehensible, or invalid strategy
EM = economically marked
OM = overly marked
CM = cut as marked
CE = cut economically
Preserved-pieces strategies

Economically marked
Cut as marked

Overly marked
Cut economically

Overly marked
Cut as marked

Mark-all strategies

Economically marked
Cut as marked

Overly marked
Cut economically

Overly marked
Cut as marked

Distribution strategies

Economically marked

Overly marked

Figure 5. Children’s partitioning strategies for continuous, dissectible, like elements.
Partitioning Wholes Consisting of Continuous, Dissectible, Unlike Elements

Table 3 shows student strategies for partitioning composite wholes with unlike elements, such as pizzas and Chinese dinners. In contrast to the strategies used in partitioning four like pizzas (the four pepperoni pizza pies), when the pizzas all had different toppings, students favored a distribution strategy. It appears that they cognitively differentiated between the two situations on the basis of the like/unlike characteristics of the items composing the whole. When partitioning Chinese dinners, although the numbers in the four-meals problem matched those in the pizza problem, more students used the distribution strategy than with the pizzas. Probably because we customarily share Chinese dinners, there was less concern for economy and a clear concern that each person receive a portion of each dinner. In dividing three meals among six people, strategies were apparently more strongly influenced by the numbers than by social custom; that is, students may have used the most economical partition because the sharing could be so easily accomplished by cutting the meals in half. Figure 6 shows students’ distribution strategies for each of the three problems.

Table 3
Percentage of Students in Each Grade 5–8 Using Each Partitioning Strategy for the Four-Pizzas, Three-Meals, and Four-Meals Problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
<th>Preserved-pieces strategy</th>
<th>Mark-all strategy</th>
<th>Distribution strategy</th>
<th>In</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>EM-CM</td>
<td>OM-CM</td>
<td>OM-CM</td>
<td>EM-CM</td>
</tr>
<tr>
<td>4 Pizzas</td>
<td>4</td>
<td>14</td>
<td>9</td>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>20</td>
<td>2</td>
<td></td>
<td>28</td>
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<td></td>
<td>6</td>
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<td>4</td>
<td>8</td>
<td>10</td>
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<td></td>
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<td>4</td>
<td>4</td>
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<tr>
<td></td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3 Meals</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>75</td>
</tr>
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<td></td>
<td>5</td>
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<td>69</td>
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<td></td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>88</td>
</tr>
<tr>
<td>4 Meals</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td></td>
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<td></td>
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<td></td>
<td>8</td>
<td>3</td>
<td></td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\[n_4 = 63; n_5 = 60; n_6 = 72; n_7 = 69; n_8 = 82\]

In = incomplete, incomprehensible, or invalid strategy
EM = economically marked
OM = overly marked
CM = cut as marked
CE = cut economically

Only two of the children who were interviewed, one sixth grader and one seventh grader, standardly applied distribution strategies to all the pizza (like and unlike items) and Chinese dinner problems. As is illustrated in Andy’s discussion of the
Four unlike pizzas for three people

Three unlike Chinese dinners for six people

Each person gets a sixth of each meal or the get 1/2 of the total food.

Four unlike Chinese dinners for three people

Figure 6. Some common distribution strategies for partitioning composite wholes composed of unlike elements.
pepperoni pizza problem, a striking feature of their interviews was their inability to distinguish amount of pizza from number of pieces of pizza:

I: How much pizza does one person receive?
A: Well you can see each one would get eight pieces. I gave out the pieces by twos; then they each got another two slices and another two; then there was enough left to give everybody two more slices, so they each got eight.
I: How much of the total amount of pizza did one person eat?
A: Eight slices.
I: How much is eight slices?
A: A lot.
I: Is it more than a whole pizza?
A: Yeah. If you put all the slices on one pan, it would be more than one.
I: Can you think of a way to tell me how much pizza it would be? But tell me without talking about slices.
A: Eight … um … wedges, triangles like?
I: Andy, suppose I had already cut the pizzas into three pieces each instead of six. How much pizza would each person get?
A: (Redraws, marks, and distributes.) Four pieces.
I: Is that the same amount of pizza he got last time?
A: Maybe. Well, no. It’s less.
I: Suppose you don’t know how many slices the pizza man cuts into each pizza, but you want each person to get the same amount of pizza. How much would you tell him to give each person?
A: Probably enough slices to fill up a pan plus one or two more.

Among the children who differentiated between the two food types in their partitioning strategies, it was clear that they cognitively differentiated between the problems as well. During his interview, Robert, an eighth grader, struggled to express the conceptual difference in sharing like pizzas and unlike Chinese dinners:

I: How much of the Chinese food does one person receive?
R: Well, like I said, 1/6 of each.
I: But how much of the total amount of Chinese food is that?
R: (Long pause) I don’t know what you mean.
I: Look back to the pizza problem you did a minute ago. After you shared the four pepperoni pizzas among the three people, you told me the amount that each one would receive: 1 1/3 pizzas. You said that each person would get 1 1/3 of the pepperoni pizzas you started with. In this problem, can you tell me the amount of Chinese food each person should receive?
R: (Long pause) I don’t think so. I mean … it doesn’t make sense. I mean if you shoved together …. Look, let’s put it this way. If you took all the one-sixths that belong to one person, you would get four. If you shoved them all into one of those little white boxes, you could like fill it up four sixths of the way up, but its not four sixths of one of the dinners, because you wouldn’t mix ‘em all up like that in one box. See what I mean?
I: Yes, I think I do.
R: Yeah. I think it’s a tricky question because that stuff isn’t four sixths of anything. They’re all getting the same amount of food, but it doesn’t seem right to call it four sixths of the dinners.
I: What do you mean?
R: Well, you couldn't say it's four sixths of all the dinners because you would never mix them up either. Right? The only way you can say it is one sixth of each dinner. You can't add 'em up.

**Partitioning Composites of Discrete Items**

Table 4 shows the percentage of students using each strategy when partitioning composites of discrete items, such as packs of gum or six-packs of beverages, and Figure 7 illustrates students’ strategies for the cola problem. Although the results show an increasing ability from fifth through eighth grade to use economical strategies, fewer students used preserved-pieces strategies than used mark all strategies. If we compare strategies with, say, the pepperoni pizza problem, in which many students were able to give each person a whole pizza without marking the individual pieces of the pizza, results on the gum and cola problems seem to indicate that students had a stronger need to see individual items when using prepackaged discrete elements to be sure that each person was getting the same number of pieces. In solving these problems, only a few students used a distribution strategy, drawing 25 sticks of gum or 48 cans of cola. Rather, in the dominant strategy, the composite items were opened to reveal the contents but not unpacked; for example, the packs of gum that did not require splitting were drawn with 5 pieces sticking out. This may indicate that for those students, the pack of gum was not yet a conceptual composite; they were not yet ready to think about a pack as an entity apart from its constituent pieces.

The importance of the perceptual support for composite units characterized the interviews of students who used mark-all strategies for the gum and cola problems. Sheree, a sixth grader, used the mark-all strategy in both cases. In her interview, when she talked about her strategy, it was clear that although she was beginning to think in terms of a composite unit, she was still counting pieces.

I: Sheree, I noticed that here you drew in all the little sticks of gum in each package. Why did you do that?
S: I just wanted to make sure each person got the same amount.
I: Hold on a moment, I want to get something out of my purse. OK. I have two new rolls of life savers. If I give you one roll and I keep one roll for myself, do you think we both have the same amount of candy?
S: Yes.
I: How do you know? I didn’t see you count the lifesavers.
S: I just know it.
I: Then why couldn’t you just know it when you drew the packs of gum?
S: Because, look. (S points and counts the individual sticks of gum in one person’s share.) There’s a little piece and one, two, three, four—now what? (pointing to a pack) Do I say five? No. I have to count what’s inside: five, six, seven, eight, nine.

Sheree’s concerns stood in contrast to remarks made by Tom, one of the seventh-grade students, as he justified his preference for the preserved-pieces strategy.

T: It’s too boring to draw all those little sticks of gum. You know they’re in there. You don’t need to waste time drawing them.
Solutions to the juice problem reflected the influence of given numbers and the given composite items. Because students were given two 6-units for four people, many of them designated a share as half a six pack. In contrast, in the cola problem, no one used half a six pack and one can, but instead, gave each person four cans.

Cutting Arrays and Sectioned Wholes

The percentages in Table 5 indicate that student strategies tend toward economy even in the case when they are cutting arrays and prepartitioned items. Although these items might present constraints or tend to suggest certain cutting patterns and discourage others, most student strategies reflected an effort to use as few cuts as possible. Greater economy was shown in the egg problem, probably because of the numbers. Students cut the array so that each person received a 4-unit or two 2-units. The candy problem was more difficult, because even with the most economical cutting, one of the shares required a piece of each bar.

Figure 8 shows six of the strategies elicited by the candy problem. The first strategy was typical of the older students who were concerned about making too many cuts, again because of the loss of crumbs. The optimal strategy requires only four cuts to separate the shares. It reflects an effort to create the greatest number of 4-units and cannot be achieved by counting out blocks of four in a systematic way. The other strategies, beginning with the second, require 5, 6, 6, 8, and 10 cuts, respectively. In each
of the second, third, and fourth strategies, it appears that the student knew that a share consisted of four rectangles and proceeded to count by fours, covering the blocks of candy in an orderly fashion without regard for maintaining a composite 4-unit. In the last two strategies, the students were meting out portions in terms of 1-units and 2-units, respectively.

Preserved-pieces strategy

Mark-all strategy

Distribution strategy

Figure 7. Common strategies used to partition the cola problem (a composite whole).
Table 5
Percentage of Students in Each Grade 5–8 Using Each Partitioning Strategy for the Eggs and Candy Problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
<th>Distribution Strategy</th>
<th>In</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>EM</td>
<td>OM</td>
</tr>
<tr>
<td>Eggs</td>
<td>4</td>
<td>32</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>58</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>57</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>75</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>76</td>
<td>14</td>
</tr>
<tr>
<td>Candy</td>
<td>4</td>
<td>19</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>40</td>
<td>43</td>
</tr>
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<td></td>
<td>7</td>
<td>46</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>55</td>
<td>32</td>
</tr>
</tbody>
</table>

\(n_4 = 63; n_5 = 60; n_6 = 72; n_7 = 69; n_8 = 82\)

In = incomplete, incomprehensible, or invalid strategy
EM = economically marked
OM = overly marked

Figure 8. Student strategies for cutting the candy bars.
DISCUSSION

The results of this study characterize children's unitizing process as one that develops over time and with experience in varied contexts. Decomposition of a given whole into small units appears to happen immediately and naturally in the course of fair-share activities, but reunitization into composite pieces, shown operationally as greater economy in marking and cutting fair shares, develops less rapidly. In this study, students engaged in intuitive activities in which they experienced amount, not merely number; the relevant question switched from "How many?" to "How much?" and their partitioning strategies illuminated some of the stages they went through as they applied and extended the number sense and counting processes that had served them through whole number concepts and operations in order to make sense of more complex operations that generate quantity.

This study also highlights the subtle and often tacit interaction of context and cognition as it affects the shape of one's mathematics. It showed that student partitioning strategies were situationally specific and showed a strong observance of social practice and practicality.

In short, for these students, the tasks were modeling activities, dependent on both external information and the internal cognitive models the students had constructed from their past experiences. The students made decisions about which aspects of the real world they would attend to (cutting produces loss), they decided how to measure quantities (choice of units), they sometimes refined their initial judgments (cut into thirds after marking sixths), and made attempts to interpret their results in terms of given information (Is that how much of the original amount of pizza each person gets?).

Across grade levels, evidence showed that children's cognitive models developed from primitive to more refined states. Because the teacher's ultimate goal is that students should eventually construct the highly idealized and invariant numerical model common to all situations in which, for example, four 1-units are divided by three, it is important to understand children's models and how they develop, so that instructional decisions can serve to facilitate student construction of increasingly abstract ideas. Thus, this study suggests new perspectives on partitioning as a didactic activity. Who needs partitioning activities? Why? How much is enough? What are the signals of improvement in a student's partitioning process?

Trends Related to Unitizing

The large percentages of decompose-and-distribute strategies applied to the partitioning problems attest to the fact that children have little trouble reunitizing into smaller pieces. Decomposition, however, is not an immediately reversible operation. The reunitization of a quantity in terms of more composite units appears to be a complex process that happens in stages. Although there may be some basis for using M-space\(^1\) arguments (Case, 1985) to explain why the younger children in the

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\(^1\)Since the size of working memory is limited, problem-solving strategies cannot become more sophisticated until the current strategy becomes so automated that several operations are chunked into one, thus freeing space in the working memory.
study used distribution strategies more often than the older children did, the formation of composite units seems to involve an interaction of other mathematical ideas. For example, number sense plays a role in creating the difference between the EM and OM conditions under the distribution strategy. The step from OM to EM occurs because children either get tired of distributing large numbers of singletons one at a time, or their number sense allows them to anticipate that they will not exhaust the supply of pieces if they distribute more than one at a time.

The mark-all strategies represent an intermediate step in the construction of composite units in which students still need some degree of visual assurance that each share will have the same number of pieces. Students who use the mark-all strategy are beginning to group singleton units, but they can think about the new grouping only with perceptual support of its constituent parts, much the same way that a younger child’s concept of “fiveness” may depend on her ability go back to and count five fingers on her hand. For these students, a whole pizza was not yet conceptually a composite unit; it was six \( \frac{1}{6} \)-units. It was evident in the student protocols that Sheree had not yet conceptually formed a composite unit, whereas Tom no longer needed visual support.

Within the PP and MA strategies, the role of visual cues in mediating the step to unit composition is further confirmed by considering the existence of the OM-CE substrategies. Instead of cutting and distributing the pieces as they have marked them, the students were able to see that, for example, they could group two one-sixth pieces of pizza and cut as if they had marked thirds. In the end, a more composite unit was used than the student may have intended at first because of a visually induced operational equivalence. Students were “experiencing” equivalent amounts.

**The Importance of Paper-and-Pencil Activity**

The OM-CE strategies suggest that paper-and-pencil tasks may be important supplements to the physical acts of sharing in early childhood for the purpose of facilitating the development of more composite units. In the graphical representation, students had the option of cutting food into larger pieces than they had marked. In real life, breaking or cutting may be less precise, and food items are rarely marked in advance. Thus, there is a greater chance that notions of equivalence will be visually induced in the paper-and-pencil tasks. The use of mark-all strategies, a widely used intermediate step supporting the conceptual development of more composite units, is simply not available to the student in physical sharing activities.

Other advantages seem to accompany the drawing tasks as well. In physically sharing four cookies among three children, for example, one might deal out one cookie to each person and then forget about those three cookies. The task is then modified to one of sharing one cookie among three people. Although the person doing the sharing may be reasonably confident that this approach is valid—a “theorem-in-action,” as Vergnaud (1988) might call it—the total amount that each person receives is not a salient feature of the process and may not be conceptually chunked into a single quantity. In the graphical representation, the total share remains conspicuous for discussion and consideration as a quantity.
Children’s Notion of Quantity

Accompanying this process of forming composite units is a corresponding change in the meaning of the word “amount” as the students talk about their work. Of paramount importance to the students who use distribution strategies is that each person get the same number of pieces. Andy’s and Sheree’s interviews indicated that these students were using the word “amount” in a counting sense. Robert clearly used the word to denote quantity. It appears that after the student reaches the point where he can see sameness in terms other than numbers, the conceptual adjustment is reflected in the way the word “amount” is used. Thus, partitioning activities have provided us with visual and verbal evidence that children are making an important conceptual leap. They are beginning to bridge the gap between additive and multiplicative reasoning with the use of equivalence mechanisms. For some students, in particular, those who marked smaller pieces than they actually cut, equivalence was probably visually induced. For others, the adoption of larger composite units was due to their understanding of a quantitative notion of equivalence: the inverse relationship between the size and the number of pieces and the invariance of amount under these transformations.

Robert’s sophisticated discussion of the Chinese dinner problem indicated that he had some clear notions about when you can and cannot add and that he understood quantities were more than numbers divorced from referents. Sometimes an addition results in a sum whose referent is different from the referents of both of the addends, as in the case of 2 apples + 3 oranges = 5 pieces of fruit. However, Robert was grappling with a more abstract, algebraic notion of quantity: $2a + 3b$ does not give $5a$ or $5b$ or 5 of anything.

At the same time, however, even Robert, one of the more sophisticated thinkers, lacked some deeper understandings and distinctions about the quantities he was constructing. After sharing four pizzas among three people, he said that each would receive $1\frac{1}{3}$ pizzas and, on further probing, said that each would receive $1\frac{1}{3}$ of the total amount of pizza. In fact, each person receives $1\frac{1}{3}$ pizzas, or $\frac{1}{3}$ of the original amount. It is not clear that Robert grasped this important distinction, and thus he was unable to attach the proper interpretation to his results.

Implications for Instruction

Partitioning has not been fully exploited as a didactic device for helping children to develop rational number ideas. For the teachers whose classes were involved in this research, either partitioning was a foreign notion, or it was considered a third-grade introductory fraction activity. This study strongly invites partitioning activities into the middle school curriculum and encourages their sustained use until students have attained economical strategies across a wide variety of contexts. In particular, partitioning is more than an introductory fraction activity. There are obviously many strategies that can be used to specify the amount in a fair share, but it is not sufficient merely to observe that students can use one of them to obtain a correct result; partitioning strategies increase in sophistication. Certainly many students,
in particular the older ones, would have known before engaging in the partitioning activity that four pizzas shared among three people would give $1\frac{1}{3}$ pizzas per person, but their task here was to produce the share graphically, not symbolically. The wide range of strategies within and across grade levels showed that students know $1\frac{1}{3}$ pizzas in many different ways.

Although the present study provides knowledge about performance rather than about learning, its cross-sectional design supports inferences about growth in the unitizing process. Armed with some baseline knowledge about children’s preinstructional partitioning strategies and a framework for defining a desirable direction for the evolution of ideas, the skillful teacher can harness children’s useful ideas and informal knowledge and direct them through carefully chosen situations to facilitate development. For example, when a student persistently uses a mark-all strategy for packs of gum, increasing the number of packs in each partitioning situation may cause the student to question the need to draw and count the five sticks in each and every pack, or if a student uses an inefficient distribution strategy, questions related to economy may encourage more sophisticated strategies: Is there a better way to do that? Could you think of a way to do it using fewer cuts? In general, the same types of problems used to assess children’s ability in unitizing and partitioning could be learning tools if incorporated into the curriculum in a systematic way. Table 6 summarizes the dimensions and directions that might be incorporated into instructional activities to promote increasingly sophisticated partitioning strategies.

<table>
<thead>
<tr>
<th>Dimensions of growth</th>
<th>Directions of movement</th>
<th>Didactic methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variety</td>
<td>Partitioning a limited number of task types to partitioning in a variety of contexts</td>
<td>Incorporate all of the continuous and discrete problem types in Table 1</td>
</tr>
<tr>
<td>Broad conception of the whole</td>
<td>Simple wholes to wholes composed of multiple pieces to composite wholes</td>
<td>Vary the nature of the given whole: simple to complex composite</td>
</tr>
<tr>
<td>Invariance of amount</td>
<td>How many to how much</td>
<td>Have students compare multiple solutions to see different numbers of pieces but invariance of amount of food; for various solutions, discuss how much in a share and how much as compared to the original amount</td>
</tr>
<tr>
<td>Economy</td>
<td>Distribution strategies to mark-all strategies to preserved-pieces strategies</td>
<td>Challenge students to complete a partitioning task using the smallest number of cuts; increase size of numbers</td>
</tr>
<tr>
<td>Flexibility in unitizing</td>
<td>Finding a share in terms of a single unit to reformulating that amount in terms of other units</td>
<td>Have students find several equivalent solutions; vary numbers so that shares consist of $&gt;1$ and $&lt;1$ piece of the whole; choose numbers so that solutions involve more than halving and doubling; discuss the conditions under which one solution might be better than another</td>
</tr>
</tbody>
</table>
The study further highlights the importance of helping children to think and speak accurately about the shares they are constructing. Even when they have partitioned correctly, they need to reexamine their results and interpret them in terms of the initial quantity. To encourage the multiplicative interpretation, the question “How much?” should be used to prod the student who prefers to count pieces toward relative thinking. One share consists of how much pizza? What part of the total amount of pizza is that share?

The many personalities or subconstructs of rational number that children must conceptually coordinate may all be understood as compositions and recompositions of units. Because the rational numbers are a quotient field, partitioning itself is an operation that plays a role in generating each of those subconstructs. This suggests the importance and complementary nature of unitizing and partitioning. Students need extensive presymbolic experiences involving these conceptual and graphical mechanisms in order to develop a flexible concept of unit and a firm foundation for quantification, to develop the language and imagery needed for multiplicative reasoning, and to conceptually coordinate the additive and multiplicative aspects of rational numbers. It remains a challenge to researchers who do theoretical analyses of content and to those who do empirical studies to continue to identify critical concepts and operations and to suggest promising didactic approaches.

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